### Optimization for partial differential systems

**Homework**

### Task 1. Transformation inverse problems to optimization problems

Consider heat transfer phenomenon. The mathematical model of the system is the heat equation with boundary conditions



where *u* is the temperature,

*t*  is the time,

*x* is the spatial variable,

*L* is the length of the body,

*ρ* is the density,

*c* is the heat capacity,

*λ* is the thermal conductivity,

*f* is the heat source,

*ϕ* is the initial temperature,

*a* is the boundary temperature in the left,

*b* is the boundary temperature in the right.

**Variants**

|  |  |  |
| --- | --- | --- |
| **variant** | **unknown parameter** | **measurable information** |
| **1** | ***f*** |  |
| **2** | ***a*** |  |
| **3** | ***λ*** |  |
| **4** | ***c*** |  |
| **5** | ***b*** |  |

**The inverse problem**: in is necessary to find the unknown parameter such that the solution *u=u*(*x*,*t*)of the system satisfies measurable information condition.

**It is necessary** to transform the given inverse problem to the corresponding optimization problems.

### Task 2. Function minimization

|  |  |  |
| --- | --- | --- |
| **Variant** | **Question 1** | **Question 2** |
|  | Use the stationary condition for the concrete function *f.*  Check the properties of the stationary points. | Choose the function with the given property. |
| 1 |  | The stationary condition has a unique solution that is not a point of the minimum. |
| 2 |  | The stationary condition does not have any solutions. |
| 3 |  | The stationary condition  is the necessary and sufficient of the minimum. |

**Remark**. Do not use the examples from the lecture. This will not count.

### Task 3. Functional minimization

Find the Gateaux derivatives from the given functions of two variables and integral functionals.

|  |  |  |
| --- | --- | --- |
| **variant** | **function** | **functional** |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |

**Remark**. The variant number is chosen in the same way as in the previous tasks.

### Task 4. Functional minimization. 2.

For the examples from Task 3 (function of two variables and integral functional) write the gradient method.

### Task 5. Bounded minimization problems

**Question 1**. **Variational inequality**

It is given the minimization problem for functional



on the set of functions *v* such that *c*≤*v*(*x*)≤*d* for all *x*∈[*a*,*b*]. It is necessary find its solution using variational inequality.

**Variants**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **variant** | ***z*(*x*)** | ***a*** | ***b*** | ***c*** | ***d*** |
| 1 | *x*3 | -2 | 2 | -1 | 1 |
| 2 | *-x*3 | -1.5 | 1.5 | -1 | 1 |
| 3 | -2*x* | -1 | 1 | -1 | 1 |

**Question 2**. **Gradient method**

It is necessary to choose the function of two variable from your tasks 3 and 4. You minimize it on the set, where first argument belongs to the interval [0,1], and second one belongs to the interval [0,2]. It is necessary to write the formula of the project gradient method.

### Task 6. Abstract inverse problems. Adjoint operators

Find the adjoint operator for the given operator on the space of the smooth functions with zero values on the boundary of the given interval [0,1]. It is necessary to use the definition of adjoint operators. The scalar product here the integral of product of the considered functions.

1. 

2. 

3. 

### Task 7. Inverse problems for ordinary differential equations

It is given the inverse problems

Variant 1. 

Variant 2. 

Variant 3. 

It is necessary to do the following:

1. Write the corresponding optimization problem
2. Find the functional derivative
3. Write the gradient method

Remark. These inverse problems are the partial cases of the final lecture

### Task 8. Inverse problems for ordinary differential equations

It is given the inverse problems with unknown parameter *v.*

Variant 1. 

Variant 2. 

Variant 3. 

It is necessary to do the following:

1. Write the corresponding optimization problem
2. Find the functional derivative
3. Write the gradient method

### Task 8. Inverse problems for the boundary problems for second ordinary differential equations

We consider boundary problem for the one-dimensional elliptic equation that is a boundary problem for a second order ordinary differential equation. The boundary value *v* is unknown. The additional information is given on another boundary: derivative of the solution is given there.

It is necessary to do the following:

1. Write the corresponding optimization problem
2. Find the functional derivative
3. Write the gradient method

**Variants**

1) 

2) 

3) 

### Task 10. Inverse problems for the heat equation

We consider boundary problem for the one-dimensional heat equation with unknown function *v*. It is necessary to do the following:

1. Write the corresponding optimization problem
2. Find the functional derivative
3. Write the gradient method

**Variants**

1) 

2) 

3) 

### Task 11. Inverse problems for the heat equation

We consider boundary problem for the one-dimensional heat equation with unknown function *v*. It is necessary to do the following:

1. Write the corresponding optimization problem
2. Find the functional derivative
3. Write the gradient method

**Variants**

1) 

2) 

3) 

### Task 12. Inverse problems for the wave equation

We consider boundary problem for the one-dimensional wave equation with two unknown function *v*1 and *v*2. It is necessary to do the following:

1. Write the corresponding optimization problem
2. Find the functional derivative
3. Write the gradient method

**Variants**

1) 

2) 

3) 

### Task 13. Well-posedness of problems

It is necessary to prove the Tikhonov ill-posedness for the following optimization problems, using the corresponding example form lecture as a sample. The analysis includes the following steps:

1. Determining of the solution of the problem that is trivial.
2. Determining a trigonometric minimizing sequence of control.
3. Proving that this sequence does not converge to the solution of the problem.

**Variants**

Prove the Tikhonov ill-posedness for the following optimization problems:

1. 

2. 

3. 

### Task 14. Non-smooth optimization

Consider the non-smooth optimization problems. It is given the concrete function. It is necessary

1. Prove that this function is not differentiable.
2. Describe its subdifferential.
3. Describe the subgradient method.

**Variants**

1. *f*(*x*) = *x*2 + 3|*x–*1|.
2. *f*(*x*) = *x*3 + 2|*x+*1|.
3. *f*(*x*) = sin*x* *–* 2|*x+*3|.